REPRESENTATION OF THE PROBABILITY INTEGRAL IN TERMS OF AN ELEMENTARY FUNCTION

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A function is described as an approximation to the probability integral over the whole range of its definition.

In the calculation of temperature fields, various functions using the probability integral erf (u) are frequently encountered.

For example, to calculate the temperature at the center of a fixed distributed source of square shape, acting on the surface of a semi-infinite body, we have

$$\Theta_{(0,0t)} = \frac{q\sqrt{a}}{\lambda\sqrt{\pi}} \int_0^{\tau} \frac{\left[\operatorname{erf}\left(\frac{l}{4\sqrt{at}}\right)\right]^2}{\sqrt{t}}\, dt. \quad (1)$$

As is known, integrals of this type do not reduce to the functions investigated and must be evaluated by some approximate method.

The approximate calculation in the above and other similar cases is considerably facilitated if we set

$$\operatorname{erf}(u) = \pm\sqrt{1-\exp(-1.26u^2)}\,. \quad (2)$$

For $u > 0$ we should take the plus sign in front of the radical, and for $u < 0$, the minus sign. It is easy to verify the validity of this substitution from an examination of Table 1.

An investigation of the nature of the functions erf (r) and the substitution function $\pm(1 - \exp(-1.26\mu^2))^{1/2}$, by determining the signs of the second derivatives shows also that the graphs of both functions are convex in the region of definition $(0,+\infty)$. The numerical values of the second derivatives of these functions are quite close over the whole range of definition.

Thus, the function $\pm(1 - \exp(-1.26\mu^2))^{1/2}$ is a fairly accurate substitute for the probability integral.

Using (2), we will demonstrate solution of the integral in Eq. (1). Using the substitution $1.26l^2/16a = b$, and setting $b/t = x^2$, we obtain, following substitution,

$$\int_0^{\tau} \frac{\left[\operatorname{erf}\left(\frac{l}{4\sqrt{at}}\right)\right]^2}{\sqrt{t}}\, dt \cong$$

$$\cong 2\sqrt{\tau} - 2\sqrt{b} \int_{\sqrt{b/\tau}}^{\infty} \frac{\exp(-x^2)}{x^2}\, dx.$$

Using the solution of [2],

$$\int_u^{\infty} \frac{\exp(-p^2x^2)}{x^{2n}}\, dx =$$

$$= \frac{(-1)^n 2^{n-1} p^{2n-1}\sqrt{\pi}}{(2n-1)!!}[1-\operatorname{erf}(pu)] + \frac{\exp(-p^2u^2)}{2u^{2n-1}} \times$$

$$\times \sum_{k=0}^{n-1} \frac{(-1)^k 2^{k+1}(pu)^{2k}}{(2n-1)(2n-3)\ldots(2n-2k-1)};$$

with p = 1, n = 1, we obtain

$$\int_0^{\tau} \frac{\left[\operatorname{erf}\left(\frac{l}{4\sqrt{at}}\right)\right]^2}{\sqrt{t}}\, dt =$$

$$= 2\sqrt{\tau} - 0.56\frac{l}{\sqrt{a}} \times$$

$$\times \left\{ \frac{\exp\left(-\frac{1.26l^2}{16a\tau}\right)}{\frac{0.28l}{\sqrt{a\tau}}} - \sqrt{\pi}\left[1 - \operatorname{erf}\left(\frac{0.28l}{\sqrt{a\tau}}\right)\right] \right\}. \quad (3)$$

Table 2 shows a comparison of values calculated by graphical integration with values from Eq. (3), with $a = 0.83\ \text{cm}^2/\text{sec}$ and $l = 0.005$ cm. The specific values

Table 1

Comparison of Values of the Probability Integral and the Substitution Function

u	erf (u)	$\sqrt{1-\exp(-1.26u^2)}$	u	erf (u)	$\sqrt{1-\exp(-1.26u^2)}$	u	erf (u)	$\sqrt{1-\exp(-1.26u^2)}$
0	0	0	0.2	0.2227	0.2218	1,3	0.9340	0.9427
0.01	0.01128	0.01122	0.3	0.3286	0.3274	1,4	0.9523	0.9568
0.02	0,0226	0.02247	0,4	0.4284	0.4272	1.5	0.96611	0.9672
0.03	0.0338	0.3367	0,5	0.5205	0.5198	1,6	0.9763	0.97995
0,04	0.0451	0.04494	0,6	0,6039	0.6038	1.7	0.9837	0.9854
0,05	0.05637	0.05612	0,7	0.6778	0.6787	1,8	0.9891	0.9915
0,06	0.0676	0.06734	0.8	0.7421	0.744	1.9	0.9928	0.9958
0.07	0.0789	0.07855	0.9	0.7969	0.79965	2	0.99532	0,9967
0,08	0.0901	0.08979	1,0	0,8427	0.8464	2,5	0.9996	0.9998
0,09	0,1013	0,101	1,1	0.8802	0.8850	3	0.99998	0.9999
0.10	0,11246	0,1119	1.2	0.9102	0.91516	∞	1	1

Table 2
Value of the Integral in Equation (1)

τ, sec	Graphical determination	Calculation from Eq. (3)
0	0	0
10^{-6}	0.00197	0.001971
$5\cdot10^{-6}$	0.00347	0.00346
10^{-5}	0.00396	0.00404
$5\cdot10^{-5}$	0.00474	0.00486
10^{-4}	0.00495	0.00493

taken relate to calculation of temperature in a single grain during diamond polishing.

The probability integral representation given by Eq. (3) can be recommended also for other computational cases.

NOTATION

q is the heat source strength; a is the thermal diffusivity; λ is the thermal conductivity; l is the length of a side of the heat source; τ is the time of application of source; t is variable time.

REFERENCES

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